BENDING OF A CIRCULAR PLATE ON A NONUNIFORM ELASTIC FOUNDATION IN PARTIAL CONTACT WITH THE PLATE

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Problems concerning the design of structures on an elastic foundation are of great practical importance. By choosing an elastic isotropic half-space as the design model, many investigators have constructed solutions to problems of the bending of plates for a wide range of plate loads and shapes [1-6]. However, the use of these solutions in engineering practice has revealed several shortcomings of the model of a uniform elastic half-space. In particular, the theoretical values of deflection and bending moment are overstated. One reason for this is the nonuniformity of the elastic properties of most actual foundations (such as soils) through their depth. Attempts have been made to allow for this nonuniformity and correct the results obtained from the uniform elastic half-space model by also examining an elastic model which is an exponential or power function of depth (here, the Poisson's ratio is assumed to be constant). An exhaustive listing of the published investigations of this problem can be found in the bibliographies of [7-9], while a survey of individual studies can be found in [4, 5, 10, 11].

Here, we examine an axisymmetric problem concerning the bending of a circular plate on an elastic foundation with which it is in incomplete contact (with the condition of unilateral constraint). The model of the foundation is an elastic isotropic continuous-nonuniform half-space with a depthwise-varying Poisson's ratio and a constant shear modulus. Solution of the problem by the method of pairwise integral equations reduces to a linear Fredholm integral equation of the second type.

1. We will examine an elastic isotropic continuous-nonuniform half-space as the elastic foundation. The points of the half-space belong to the region $z \geq 0$ in the cylindrical coordinate system r, θ, z. The shear modulus of the material of the foundation μ is assumed to be constant, while the Poisson's ratio $\nu = \nu(z)$ is an arbitrary (sufficiently smooth) function of depth. The proposed model can be regarded as a model of an elastic foundation with a variable elastic modulus. In this case, the elastic modulus changes with depth according to the familiar relation $E(z) = 2\mu[1 + \nu(z)]$.

Let a circular plate of radius R be at rest without friction on the surface $z = 0$ of the half-space. The symmetry axis of the plate coincides with the z axis. An axisymmetrically distributed normal load of the intensity $q(r)$ acts on the plate from above, while the plate contour is subjected to circumferential pairs of forces with the bending moment M. The reactive (contact) pressure $p(r)$ develops under the plate in the contact region. Outside the contact area, the surface of the half-space is free of loads. Given sufficiently

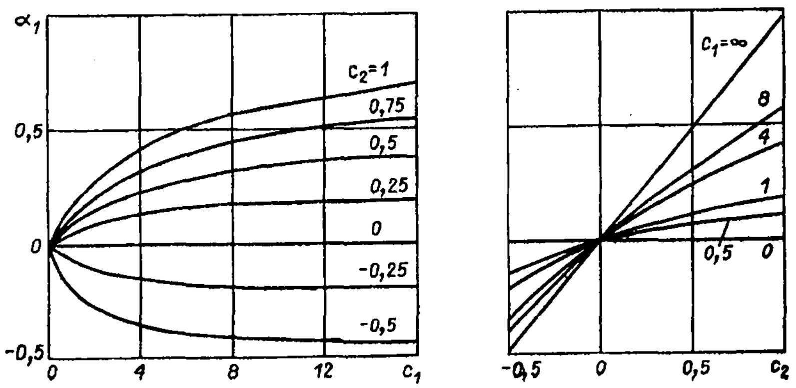

Fig. 1

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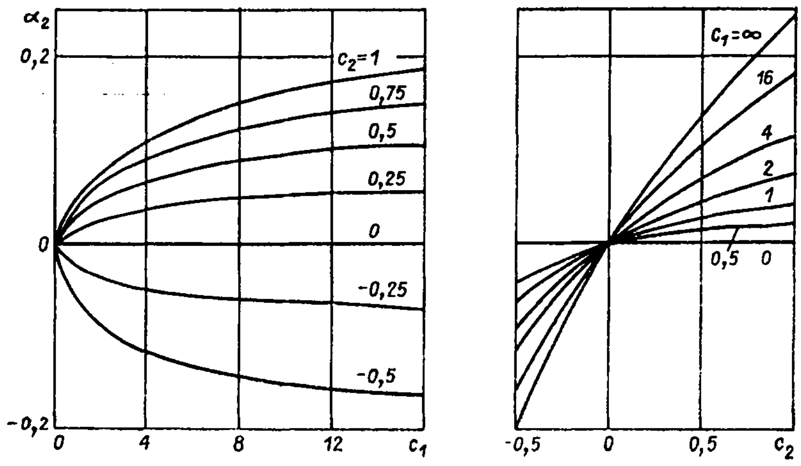

Fig. 2

small values of M, the plate will be in complete contact with the foundation. With an increase in M, the plate may separate from the foundation, i.e., the radius of the area of contact a may become less than the radius of the plate. The limiting value of the moment at which plate separation occurs will be designated as M_* (here, $a = R$).

Proceeding on the basis of the Kirchhoff hypothesis in classical plate theory, we find that the displacements of points of the middle surface of the plate (deflections) w(r) satisfy the ordinary differential equation derived by S. Germain [12]

$$\nabla^2\nabla^2 w(r) = [q(r) - p(r)]/D, \tag{1.1}$$

where $\nabla^2 = r^{-1}d[rd/dr]/dr$ is the Laplace operator; D is the cylindrical stiffness of the plate. With the prescribed plate loading conditions, the solution of Eq. (1.1) can be represented in the form

$$\begin{aligned} w(r) = w_0 - \frac{Mr^2}{2D(1+\nu_*)} + \frac{1}{4D}\int_0^r [q(t) - p(t)]\,[(r^2+t^2)\ln(r/t) - (r^2 - t^2)]\,t\,dt - \\ - \frac{r^2}{4D}\int_0^R [q(t) - p(t)]\,[\ln(R/t) + (c/2)(1 - t^2/R^2)]\,t\,dt. \end{aligned} \tag{1.2}$$

Here, $w_0 = w(0)$; $c = (1 - \nu_*)/(1 + \nu_*)$; ν_* is the Poisson's ratio of the plate material.

We can determine w_0 with the aid of the plate equilibrium condition

$$\int_0^R q(t)t\,dt = \int_0^a p(t)\,t\,dt. \tag{1.3}$$

In the absence of shear stresses at the interface, the vertical displacements of points of the boundary of the half-space $u_z(r)$ and the normal stresses at these points $\sigma_z(r)$ are connected by the relation [13, 14]

$$\begin{aligned} u_z(r) &= (2\mu)^{-1}(1 - \nu_0) r S_{-1/2,1}\{(1 + k)\psi\}(r), \\ \sigma_z(r) &= -S_{0,0}\{\psi\}(r), \end{aligned} \tag{1.4}$$

where $\psi(t) = S_{0,0}\{p\}(t)$; $p(r) = -\sigma_z(r)$ $(r \leqslant a)$; $S_{\alpha,\beta}\{f\}(t) = 2^\beta t^{-\beta}\int_0^\infty x^{1-\beta} f(x)\, J_{2\alpha+\beta}(tx)\,dx$; $k(t) = [2(1 - \nu_0)t\Lambda(t)]^{-1} - 1$; $\nu_0 = \nu(0)$; $\Lambda(t) = \int_0^\infty [1 - \nu(x)]^{-1}\exp(-2tx)\,dx$; J is a Bessel function of the first kind.

The equality of the deflections of the plate and the vertical displacements of the boundary of the half-space at points of the area of contact, together with the absence of

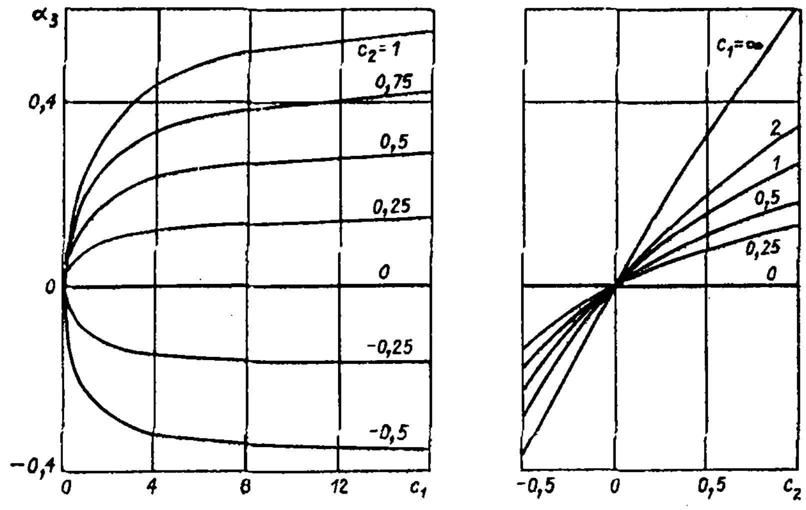

Fig. 3

loads on the surface of the half-space outside the contact area, leads to the following boundary conditions

$$u_z(r) = w(r),\ r \leqslant a,\ \sigma_z(r) = 0,\ r > a. \tag{1.5}$$

The condition of vanishing of the contact pressure at the boundary of the contact area is used to determine the radius of this area. Of the main interest in performing calculations for plates on an elastic foundation is determination of the contact pressure and the plate deflection. These quantities make it possible to then complete the design of the plate, i.e., to find the transverse force and bending moments and, thus, the normal and shear stresses in the plate.

2. Using integral representations (1.4) and satisfying boundary conditions (1.5), we arrive at paired integral equations in ψ:

$$\begin{gathered} S_{-1/2,1}\{(1+k)\psi\}(r) = 2\mu(1-\nu_0)^{-1}r^{-1}w(r),\ r \leqslant a, \\ S_{0,0}\{\psi\}(r) = 0,\ r > a. \end{gathered} \tag{2.1}$$

The normal procedure is used to reduce Eqs. (2.1) to a Fredholm integral equation of the second type [14]

$$\begin{gathered} \varphi(x) + \int_0^1 N(x, u)\,\varphi(u)\,du = H(x), \quad 0 \leqslant x \leqslant 1, \\ N(x, u) = 2\pi^{-1}\int_0^\infty k(t/a)\cos(xt)\cos(ut)\,dt, \quad H(x) = \frac{d}{dx}\int_0^x \frac{rw(ar)\,dr}{(x^2-r^2)^{1/2}}, \end{gathered} \tag{2.2}$$

where the function φ is connected with the function ψ by the relation

$$\psi(x) = 2\mu a\pi^{-1}(1-\nu_0)^{-1}\int_0^1 \varphi(t)\cos(axt)\,dt. \tag{2.3}$$

In accordance with (1.2), the right side of integral equation (2.2) depends on p(r). Using (1.4) and (2.3), we express p(r) in the right side of (2.2) through φ. After this, the integral equation takes the form

$$\varphi(x) + \int_0^1 [N(x, u) + \lambda d^3 K(x, u)]\,\varphi(u)\,du = F(x). \tag{2.4}$$

Here, $\lambda = 2\mu R^3[\pi D(1-\nu_0)]^{-1}$; $d = a/R$;

$$\begin{gathered} K(x, u) = \{(x+u)^2\ln[2(x+u)] + (x-u)^2\ln|2(x-u)| - \\ -2u^2\ln(2u) - cx^4(1-2u^2) - x^2[3 + 2c(1-d^2)u^2 - \ln d]\}/4; \end{gathered}$$

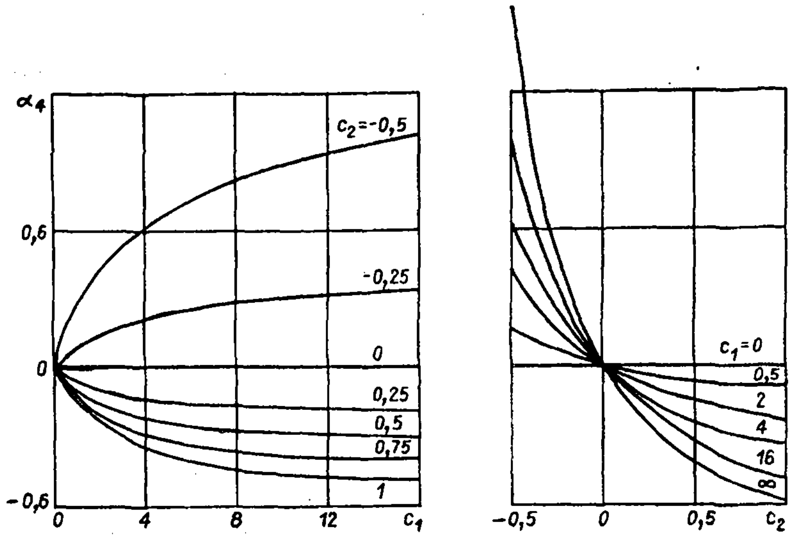

Fig. 4

$$F(x) = w_0 - Ma^2D^{-1}(1+\nu_*)^{-1}x^2 + \frac{a^4}{4D}\int_0^x q(at)\left\{(t^2+2x^2)\ln\left[\frac{x+(x^2-t^2)^{1/2}}{t}\right] - \right.$$
$$\left. - 3x(x^2-t^2)^{1/2}\right\} t\,dt - \frac{a^2R^2x^2}{2D}\int_0^1 q(Rt)\,[(c/2)(1-t^2)-\ln t]\,t\,dt.$$

Taking Eqs. (1.4) and (2.3) into account, we obtain expressions for the contact pressure and the deflections of the plate

$$p(r) = \frac{2\mu}{\pi a(1-\nu_0)}\left[\frac{\varphi(1)}{(1-\rho^2)^{1/2}} - \int_\rho^1 \frac{\varphi'(t)\,dt}{(t^2-\rho^2)^{1/2}}\right],\ w(r) = \frac{2}{\pi}\left[\int_0^\rho \frac{\varphi(t)\,dt}{(\rho^2-t^2)^{1/2}} + \right.$$
$$\left. + \int_0^1 \varphi(t)\,dt\int_0^\infty k(x/a)\,J_0(\rho x)\cos(tx)\,dx\right],\quad \rho = r/a,\quad r \leqslant a. \tag{2.5}$$

With ν = const, the function k vanishes, and Eqs. (2.4) and (2.5) give a solution of the problem of the bending of a circular plate on an isotropic uniform elastic half-space which agrees with the solution found in [15]. It should be noted that the formula for w(r) $(r \le R)$ can be obtained by inserting the expression for p(r) (2.5) into (1.2).

The plate equilibrium condition (1.3) takes the form

$$\pi R^2(1-\nu_0)\int_0^1 q(Rt)\,t\,dt = 2\mu a\int_0^1 \varphi(t)\,dt. \tag{2.6}$$

It also follows from (2.5) that the condition of vanishing of p(r) at the boundary of the contact area is equivalent to the condition

$$\varphi(1) = 0. \tag{2.7}$$

3. We will describe the procedure for finding the radius of the area of contact for the loading of a plate by a normal load uniformly distributed over its surface, i.e., with q = const. In this case, the right side of integral equation (2.4) is written in the form

$$F(x) = w_0 + \frac{qR^4}{4D}\left[\frac{d^4x^4}{6} - \frac{(2+c)}{4}d^2x^2\right] - \frac{Ma^2x^2}{D(1+\nu_*)},$$

while the equilibrium equation (2.6) becomes

$$4\mu a\int_0^1 \varphi(t)\,dt = q\pi(1-\nu_0)R^2. \tag{3.1}$$

We represent the solution of integral equation (2.4) in the form

$$\varphi(x) = w_0\varphi_1(x) + \frac{qR^4}{4D}\varphi_2(x) - \frac{Ma^2}{D(1+\nu_*)}\varphi_3(x), \tag{3.2}$$

where $\varphi_i(x)$ ($i = 1, 2, 3$) are the solutions of the Fredholm integral equations of the second kind, respectively,

$$\varphi_i(x) + \int_0^1 [N(x, u) + \lambda d^3 K(x, u)]\,\varphi_i(u)\,du = F_i(x),$$
$$F_1(x) = 1,\ F_2(x) = \frac{d^4x^4}{6} - \frac{(2+c)}{4}d^2x^2,\ F_3(x) = x^2. \tag{3.3}$$

It should be noted that the determination of a with a prescribed value of R is equivalent to the determination of d.

Satisfying conditions (2.7) and (3.1) by means of (3.2), we obtain an expression for the maximum plate deflection in the contact region w_0 and an equation relative to d:

$$w_0 = \frac{qR^4}{4\lambda D(\varepsilon_1 - \varepsilon_3\varphi_{13})}[(2/d) - \lambda(\varepsilon_2 - \varepsilon_3\varphi_{23})]; \tag{3.4}$$

$$\frac{4M}{qR^2(1+\nu_*)}d^3 + \frac{\varepsilon_2\varphi_{13} - \varepsilon_1\varphi_{23}}{\varepsilon_1 - \varepsilon_3\varphi_{13}}d - \frac{2\varphi_{13}}{\lambda(\varepsilon_1 - \varepsilon_3\varphi_{13})} = 0$$
$$\left(\varepsilon_i = \int_0^1 \varphi_i(t)\,dt,\ i = 1, 2, 3,\ \varphi_{j3} = \varphi_j(1)/\varphi_3(1),\ j = 1, 2\right). \tag{3.5}$$

Thus, the radius of the contact area is found from the simultaneous solutions of Fredholm integral equations (3.3) and Eq. (3.5). It should be noted that the above procedure for determining the radius of the contact area is easily generalized to the case of loading of a plate by a load of the intensity q(r).

The expression for the limiting moment M_* is obtained from (3.5) with d = 1;

$$M_* = qR^2(1+\nu_*)\left\{\frac{\varphi_{13}[2 - \lambda(\varepsilon_2 - \varepsilon_3\varphi_{23})]}{4\lambda(\varepsilon_1 - \varepsilon_3\varphi_{13})} + \frac{\varphi_{23}}{4}\right\}. \tag{3.6}$$

With allowance for (3.4) and (3.5), Eq. (3.2) can be represented in the form

$$\varphi(x) = qR^4(\lambda D)^{-1}\varphi_4(x),$$
$$\varphi_4(x) = \frac{2 - \lambda d(\varepsilon_2 - \varepsilon_3\varphi_{23})}{4d(\varepsilon_1 - \varepsilon_3\varphi_{13})}\varphi_1(x) + \frac{\lambda}{4}\varphi_2(x) - \frac{2\varphi_{13} + \lambda d(\varepsilon_1\varphi_{23} - \varepsilon_2\varphi_{13})}{4d(\varepsilon_1 - \varepsilon_3\varphi_{13})}\varphi_3(x).$$

Then $p(r) = -\dfrac{q}{d}\displaystyle\int_\rho^1 \frac{\varphi_4'(t)\,dt}{(t^2-\rho^2)^{1/2}}.$

4. To obtain numerical results, we will assume that the Poisson's ratio of the material of the half-space changes with depth according to the law $\nu(z) = 1 - [A + B\exp(-2\gamma z)]^{-1}$, $\gamma \geqslant 0$. Considering that the Poisson's ratio of actual materials ranges from 0 to 0.5, we find that the ranges of the parameters A and B are the intervals [1, 2] and [−1, 1], respectively.

In this case, the function k has the form $k(t/a) = c_1c_2[c_1 + d^{-1}(1 + c_2)t]^{-1}$ ($c_1 = \gamma R$, $c_2 = B/A$). Then the kernel of the Fredholm integral equation (and, thus, its solution) depend on the dimensionless parameters c, c_1, c_2, and λ. The ranges of the parameters c_1 and c_2 are the intervals $[0, \infty)$ and $[-0.5, 1]$. The parameter c_2 allows the representation $c_2 = (\nu_0 - \nu_\infty)(1 - \nu_0)^{-1}$ (ν_∞ is the Poisson's ratio at an infinitely great depth). It follows from this that $\nu(z)$ increases with depth at $c_2 < 0$ and decreases at $c_2 > 0$.

Assuming that $c_1 = 0$ or $c_2 = 0$, we arrive at the solution of the problem for a uniform half-space with the coefficient ν_0. At $c_1 \to \infty$, the solution of the problem becomes the solution for a uniform half-space with the coefficient $\nu_0(1 + c_2) - c_2$.

To check the numerical calculations, it is necessary to assign values to the dimensionless quantities λ and c. Assigning the value of the parameter λ with fixed μ, D, and R unambiguously determines ν_0. Numerical calculations were performed for $\lambda = 1$ and $c = 5/7$.

The value of M_* was calculated from Eq. (3.6) after finding φ_i ($i = 1, 2, 3$) from (3.3) with $d = 1$. The solutions of integral equations (3.3) were obtained by the method of formulas of integration [16].

Figure 1 shows the dependence of the dimensionless quantity $\alpha_1 = M_*/M_*^0 - 1$ on the parameters c_1 and c_2. Here, M_*^0 is the limiting bending moment at which plate separation occurs in the case of the problem for a uniform half-space (with the same values of λ and c). It is evident from the data that $|\alpha_1|$ increases with an increase in $c_1(|c_2|)$ at fixed nontrivial values of $c_2(c_1)$. Considering that $\alpha_1 > 0$ at $c_2 > 0$ and $\alpha_1 < 0$ at $c_2 < 0$, we conclude that $M_* > M_*^0$ at $c_2 > 0$ and $M_* < M_*^0$ at $c_2 < 0$, i.e., for an elastic foundation whose Poisson's ratio decreases with depth, plate separation occurs at a greater value of the bending moment than in the case of an elastic foundation with a constant coefficient ν_0. Separation occurs at a lower value of the bending moment for an elastic foundation whose Poisson's ratio increases with depth.

Let us assign the bending moment $M = 1.8qR^2$ ($M \approx 1.99M_*^0$). It can be seen from Fig. 1 that at this value of bending moment, the plate separates from the foundation at any value of c_1 and c_2 within their ranges of variation.

The radius of the contact area was found (to within 10^{-4}) from the simultaneous solution of Eqs. (3.3) and (3.5) by the method of successive approximations (we took $a = R$ as the initial approximation).

Figures 2-4 show the dependences of the dimensionless quantities $\alpha_2 = a/a^0 - 1$, $\alpha_3 = w_0/w_0^0 - 1$ and $\alpha_4 = p(0)/p^0(0) - 1$ on c_1 and c_2, respectively. Here a^0, w_0^0, and $p^0(0)$ are the radius of the contact area, the maximum deflections of the plate in the contact region, and the maximum contact pressure in the case of the problem for a uniform half-space (with the same values of λ and c).

It follows from the above numerical examples that allowing for an increase (decrease) in the Poisson's ratio of the elastic foundation leads to a reduction (increase) in the theoretical values of the radius of the contact area and the maximum deflections of the plate, respectively, of up to 19.7 (24.6) and 37.8% (60.8%) and to an increase (decrease) in the theoretical values of maximum reactive pressure of up to 153.8% (59.6%).

The case of a depthwise increase in the Poisson's ratio corresponds to the model of an elastic foundation whose elastic modulus increases with depth. Such a rule is generally characteristic of the elastic modulus of soils [1].

Thus, we have solved an axisymmetric problem concerning the bending of a circular plate on a nonuniform elastic foundation in partial contact with the plate. The solution takes into account the depthwise change in the Poisson's ratio of the foundation material. The results of numerical calculations provide evidence that nonuniformity of the foundation has a substantial effect on the conditions of separation of the plate from the foundation and the theoretical characteristics describing their interaction.

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